TRANSIENT GRADIENTAL FLOW OF A CONTINUOUS MEDIUM WITH A POWER-LAW RHEOLOGICAL BEHAVIOR AND A YIELD SHEARING STRESS

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An iteration scheme is proposed for solving both the direct and the reverse problem of gradiental flow which develops in a continuous medium with a power-law rheological behavior and a yield shearing stress.

Various emulsions and suspensions with mechanical properties very different than those of an incompressible Newtonian fluid are widely used in the technological processes of petroleum extraction and petroleum chemistry, as well as in power plants and in many other branches of industry. This non-Newtonian behavior becomes even more pronounced, as a rule, when the rate of a given process is increased.

Among the mathematical models of a continuous medium suitable for describing the flow of such fluids is the rheological body which combines the properties of a power-law fluid (an Ostwald–de-Walls fluid) with those of a Shvedov–Bingham plastic [1, 2]. The rheological equation for such a medium, in the case of a one-dimensional planar flow, is

$$\tau=\left(\tau_0\left|\frac{\partial v}{\partial x}\right|^{-1}+\mu\left|\frac{\partial v}{\partial x}\right|^{n-1}\right)\frac{\partial v}{\partial x}\quad \text{at}\quad |\tau|\geqslant\tau_0, \tag{1}$$

$$\frac{\partial v}{\partial x}=0\quad \text{at}\quad |\tau|<\tau_0, \tag{2}$$

with the rheological parameter n.

The physical significance of Eqs. (1) and (2) is as follows: when $|\tau| > \tau_0$, the medium has either dilatant or pseudoplastic properties; when $|\tau| < \tau_0$, the medium moves as a rigid single body. The rheological equation written in the form (1) represents a generalization of Shul'man's three-parameter equation [3].

The development of a gradiental flow of a Shvedov–Bingham plastic (n = 1) was considered in [4-6]. Here the method of solution shown in [6] will be applied to the development of a gradiental flow of a medium with a more complicated rheological behavior.

We will consider the following problem. Let a continuous medium with the rheological behavior (1)-(2) in a flat channel with fixed walls be set in motion at time t = 0 by a pressure gradient $P = \partial p/\partial z$ which varies with time.

The equation of one-dimensional planar flow for a continuous medium with an arbitrary rheological behavior is

$$\rho\frac{\partial v}{\partial t}=P+\frac{\partial\tau}{\partial x}. \tag{3}$$

By virtue of symmetry in our case, it suffices to describe the state of the medium in the lower half of the channel $-a < x < 0$ only, with the condition $\partial v/\partial x > 0$ assumed to prevail. For the zone of viscous flow we have from (1)

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$$\frac{\partial v}{\partial x} = \frac{1}{\mu^{1/n}}(\tau - \tau_0)^{1/n},$$

and integrating this expression with respect to variable x from $-a$ to x yields

$$v = \frac{1}{\mu^{1/n}} \int_{-a}^{x} (\tau - \tau_0)^{1/n}\, dx. \tag{4}$$

under the condition of adhesion between fluid and channel wall $v(-a) = 0$.

Inserting (4) into (3) and assuming that $\tau(x, t)$ under the integral sign is differentiable with respect to variable t, we obtain for $T(x, t) = \tau(x, t) - \tau_0$:

$$\frac{\partial T}{\partial x} = -P(t) + \frac{\rho}{\mu} \cdot \frac{1}{n} \int_{-a}^{x} T^{\frac{1-n}{n}} \frac{\partial T}{\partial t}\, dx. \tag{5}$$

Integrating Eq. (5) with respect to variable x from $x = y(t)$ to x, where function $y(t)$ describes the location of the interface between the viscous zone and the quasirigid core in the stream, we arrive at the functional equation

$$T(x, t) = P(t)(y - x) + \frac{\rho}{\mu n} \int_{y}^{x} \int_{-a}^{x} T^{\frac{1-n}{n}} \frac{\partial T}{\partial t}\, dx dx, \tag{6}$$

which has been derived from the condition of existence of a quasirigid core $T(y, t) = 0$.

Writing Eq. (5) for $x = y(t)$ and using the condition of flow of a quasirigid core as a single body

$$\frac{\partial T}{\partial x}(y, t) = \frac{\tau_0}{y(t)}$$

(a step-by-step derivation of this boundary condition can be found in [5, 6]), we obtain the second functional equation

$$P(t) = -\frac{\tau_0}{y(t)} + \frac{\rho}{\mu} \cdot \frac{1}{n} \int_{-a}^{y} T^{\frac{1-n}{n}} \frac{\partial T}{\partial t}\, dx. \tag{7}$$

In dimensionless form, (6) and (7) become

$$\begin{aligned} u(x, t) &= \varphi(t)(y - x) + \frac{1}{n} \int_{y}^{x} \int_{-1}^{x} u^{\frac{1-n}{n}} \frac{\partial u}{\partial t}\, dx dx, \\ \varphi(t) &= -\frac{S}{y(t)} + \frac{1}{n} \int_{-1}^{y} u^{\frac{1-n}{n}} \frac{\partial u}{\partial t}\, dx, \end{aligned} \tag{8}$$

with $S = \tau_0/P_* a$, $u(x, t) = T(x, t)/P_* a$, the channel half-width a taken as the characteristic dimension, P_* denoting the characteristic value of the modulus of the pressure gradient, and the characteristic time $\rho P_*^{(1-n)/n} a^{(1+n)/n}/\mu$.

Since the quasirigid core occupies the entire flow region at time $t = 0$, hence $y(0) = -1$ and the correct solution to system (8) requires that

$$\varphi(0) = +S.$$

The iteration scheme for solving system (8) is sufficiently obvious in the case of either the direct or the reverse problem of a developing gradiental flow.

Direct Problem. In the direct problem, for a known function $\varphi(t) \geq S$ it is required to construct $u = u(x, t)$ and $y = y(t)$. Resolving the second of Eqs. (8) with respect to $y(t)$, we easily obtain the iteration scheme:

$$u_{k+1}(x, t) = \varphi(t)(y_k - x) + \frac{1}{n} \int_{y_k}^{x} \int_{-1}^{x} u_k^{\frac{1-n}{n}} \frac{\partial u_k}{\partial t}\, dx dx, \tag{9}$$

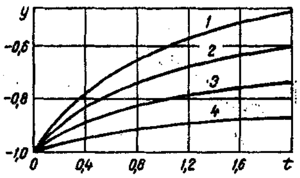

Fig. 1

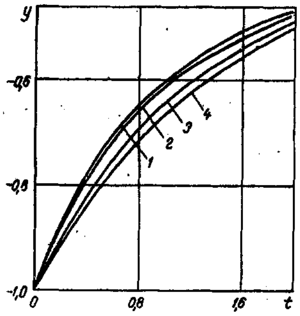

Fig. 2.

Fig. 1. Qualitative pattern of convergence of the iteration process for the direct problem with n = 0.5 and $\alpha = 1$: $y_2(t)$ curves 1, 2, 3, 4) calculated for S = 0.2, 0.4, 0.6, and 0.8, respectively.

Fig. 2. Effect of the power-law exponent n on the characteristics of planar flow with $\alpha = 1$ and S = 2: $y_2(t)$ curves 1, 2, 3, 4) calculated for n = 0.3, 0.5, 0.8, 1.0, respectively.

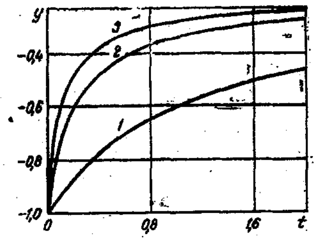

Fig. 3. Effect of parameter α on the characteristics of flow in a flat channel, with n = 0.3 and S = 0.2: $y_2(t)$ curves 1, 2, 3) calculated for α = 1, 5, 10, respectively.

$$y_{k+1}(t) = \frac{S}{-\varphi(t) + \frac{1}{n}\int\limits_{-1}^{y_k} u_k^{\frac{1-n}{n}} \frac{\partial u_k}{\partial t} dx}.$$

As the zeroth approximation we conveniently choose the quasisteady solution

$$u_0(x, t) = \varphi(t)(y_0 - x), \quad y_0(t) = -S\varphi^{-1}(t). \tag{10}$$

Following the iteration scheme (9)-(10), we have calculated the zeroth, the first, and the second approximation. For $\varphi = \varphi(t)$ we use the relation

$$\varphi(t) = S\frac{1+\alpha t}{1+\alpha St}, \tag{11}$$

with which the transient behavior under various rates of increase of the pressure gradient could be analyzed.

The results of these calculations are shown in Figs. 1-3. The effect of the plasticity parameter S on the formation of a quasirigid core is indicated in Fig. 1; the effect of the rheological power-law exponent n on the relation y = y(t) is indicated in Fig. 2; and the effect which the rate of increase of the pressure gradient (parameter α) has on the displacement of the interface between the two flow zones, as a function of time, is indicated in Fig. 3.

Reverse Problem. In the reverse problem y = y(t) is assumed known, while functions u = u(x, t) and $\varphi = \varphi(t)$ are sought. The iteration scheme becomes here:

$$u_{k+1}(x, t) = \varphi_k(t)(y - x) + \frac{1}{n}\int\limits_{y}^{x}\int\limits_{-1}^{x} u_k^{\frac{1-n}{n}} \frac{\partial u_k}{\partial t} dx dx,$$
$$\varphi_{k+1}(t) = -\frac{S}{y} + \frac{1}{n}\int\limits_{-1}^{y} u_k^{\frac{1-n}{n}} \frac{\partial u_k}{\partial t} dx. \tag{12}$$

As the zeroth approximation we have chosen

$$u_0(x, t) = \varphi_0(y(t) - x), \quad \varphi_0(t) = -\frac{S}{y(t)}. \tag{13}$$

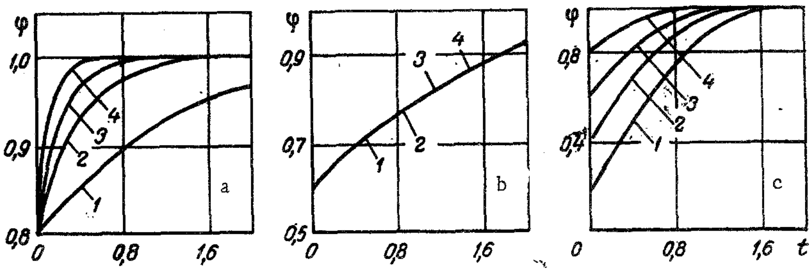

Fig. 4. Qualitative pattern of convergence of the iteration process for the reverse problem, depending on the variation of respective parameters: (a) curves 1, 2, 3, 4) calculated for $\alpha = 1, 3, 5, 10$, respectively, with $n = 0.25$ and $S = 0.8$; (b) curves 1, 2, 3, 4) calculated for $n = 0.25$, 0.7, 1.0, 1.4, respectively, with $\alpha = 1$ and $S = 0.6$ (curves 1, 2, 3, 4 for the case 4b begin to differ in the third and the subsequent places after the decimal point); (c) curves 1, 2, 3, 4) calculated for $S = 0.2$, 0.4, 0.6, 0.8, respectively, with $n = 0.5$ and $\alpha = 3$.

Following the iteration scheme (12)-(13), we have calculated the zeroth, the first, and the second approxmation. For more specific results we used the relation

$$y(t) = -[S + (S - 1)\exp(-\alpha t), \tag{14}$$

with parameter α describing the rate of increase of function $y = y(t)$.

The results of these calculations for the reverse problem are shown in Fig. 4.

The effect which the rate of increase of the pressure gradient (parameter α) has on changing the boundary condition at the channel walls, as a function of time, is indicated in Fig. 4a; the effect of the rheological power-law exponent n on the function $\varphi = \varphi(t)$ is indicated in Fig. 4b; and the qualitative pattern of convergence of the iteration process is indicated in Fig. 4c.

In conclusion, we wish to point out the regularity of the convergence pattern, equally applicable to both the direct and the reverse problem. The convergence improves, within the range of parameter values considered here ($0.2 \leq S \leq 0.8$; $1 \leq \alpha \leq 10$; $0.25 \leq n \leq 1.4$), as the value of the plasticity parameter S increases but worsens with higher values of the parameters α and n.

NOTATION

t	is the time;
ρ	is the density;
τ	is the tangential shearing stress;
τ_0	is the yield shearing stress;
τ_*	is the characteristic stress in this problem;
μ	is the dynamic viscosity;
v	is the velocity of the fluid;
x	is the transverse space coordinate;
a	is the characteristic channel dimension;
P	is the pressure gradient;
z	is the longitudinal space coordinate;
n	is the power-law exponent;
S	is the plasticity parameter;
α	is a parameter which characterizes the rate of increase of the test function;
φ	is the modulus of the pressure gradient;
P_*	is the characteristic value of the modulus of the pressure gradient;
$y(t)$	is the location of the interface between the viscous zone and the quasirigid core;
k	is the consecutive number of iterations.

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